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METRIC REGULARITY IN CONVEX SEMI-INFINITE OPTIMIZATION UNDER CANONICAL PERTURBATIONS*

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Abstract. This paper is concerned with the Lipschitzian behavior of the optimal set of convex semi-infinite optimization problems under continuous perturbations of the right-hand side of the constraints and linear perturbations of the objective function. In this framework we provide a sufficient condition for the metric regularity of the inverse of the optimal set mapping. This condition consists of the Slater constraint qualification, together with a certain additional requirement in the Karush–Kuhn–Tucker conditions. For linear problems this sufficient condition turns out to be also necessary for the metric regularity, and it is equivalent to some well-known stability concepts.

Key words. metric regularity, optimal set, Lipschitz properties, semi-infinite programming, convex programming

AMS subject classifications. 90C34, 49J53, 90C25, 90C31, 90C05

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1. Introduction.

We consider the canonically perturbed convex semi-infinite programming problem, in $\mathbb{R}^n$,

$$P(c,b): \quad \operatorname{Inf} f(x) + c'x \quad \text{s.t. } g_t(x) \le b_t, \;\; t \in T, \tag{1}$$

where $x \in \mathbb{R}^n$ is the vector of decision variables, regarded as a column-vector, $c \in \mathbb{R}^n$, c' denotes the transpose of c, the index set T is a compact metric space, $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are given convex functions in such a way that $(t,x) \mapsto g_t(x)$ is continuous on $T \times \mathbb{R}^n$, and $b \in C(T,\mathbb{R})$, i.e., $T \ni t \mapsto b_t \in \mathbb{R}$ is continuous on T.

In this setting, the pair $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ is regarded as the parameter to be perturbed. We denote by $\sigma(b)$ the constraint system associated with $P(c,b)$, i.e.,

$$\sigma(b) := \{ g_t(x) \le b_t, \;\; t \in T \}.$$

The parameter space $\mathbb{R}^n \times C(T,\mathbb{R})$ is endowed with the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with any given norm $\|\cdot\|$ and $\|b\|_\infty := \max_{t\in T} |b_t|$. The corresponding dual norm in $\mathbb{R}^n$ is given by $\|u\|_* := \max\{u'x \mid \|x\| \le 1\}$.

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Associated with the parametric family of problems $P(c,b)$, we consider the set-valued mappings $\mathcal{G} : \mathbb{R}^n \rightrightarrows C(T,\mathbb{R})$ and $\mathcal{G}^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times C(T,\mathbb{R})$ given by

$$\mathcal{G}(x) := \{b \in C(T,\mathbb{R}) \mid g_t(x) \leq b_t \text{ for all } t \in T\},$$

$$\mathcal{G}^*(x) := \left\{(c,b) \in \mathbb{R}^n \times \mathcal{G}(x) \mid x \in \arg\min\left\{f(y) + c'y \mid y \in \mathcal{G}^{-1}(b)\right\}\right\}.$$

The corresponding inverse mappings will be denoted by $\mathcal{F}$ and $\mathcal{F}^*$, respectively. Observe that $\mathcal{F}(b)$ and $\mathcal{F}^*(c,b)$ are, respectively, the feasible set and the optimal set (set of optimal solutions) of $P(c,b)$, i.e.,

$$\mathcal{F}(b) := \{x \in \mathbb{R}^n \mid g_t(x) \leq b_t \text{ for all } t \in T\},$$

$$\mathcal{F}^*(c,b) := \arg\min\{f(x) + c'x \mid x \in \mathcal{F}(b)\}.$$

Finally, by Π_c and Π_s we denote the sets of parameters corresponding to consistent or solvable problems, respectively; i.e.,

$$\Pi_c := \{(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R}) \mid \mathcal{F}(b) \neq \emptyset\}$$

and

$$\Pi_s := \{(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R}) \mid \mathcal{F}^*(c,b) \neq \emptyset\}.$$

According to Corollary 8.3.3 and Theorem 8.7 in [24], if $\sigma(b)$ and $\sigma(b^1)$ are both consistent, $\mathcal{F}(b)$ and $\mathcal{F}(b^1)$ have the same recession cone.

This paper is concerned with the metric regularity of $\mathcal{G}^*$ at a given $\overline{x}$ for $(\bar{c},\bar{b}) \in \mathcal{G}^*(\overline{x})$, that is, with the existence of neighborhoods U of $\overline{x}$ and V of $(\bar{c},\bar{b})$ and a constant $\kappa \geq 0$ such that

$$d(x,\mathcal{F}^*(c,b)) \leq \kappa d((c,b),\mathcal{G}^*(x)) \text{ for all } x \in U \text{ and all } (c,b) \in V, \tag{3}$$

where, as usual, $d(x,\emptyset) = +\infty$. In section 3 we provide a sufficient condition, (10), for this property. Essentially, it is a Karush–Kuhn–Tucker (KKT) type condition with some additional requirements.

In the particular case of linear problems of the form

$$P(c,b): \quad \text{Inf } c'x \qquad \text{s.t. } a_t'x \geq b_t, \ \ t \in T, \tag{4}$$

where $a \in C(T,\mathbb{R}^n)$ is a given function, this algebraic condition is given by (9), and it turns out to be equivalent to a condition introduced by Nürnberger [22, Condition (2) in Thm. 1.4], in relation to the stability of the strong uniqueness of minimizers (see also [11] and [13], dealing with linear optimization problems without continuity assumptions). Moreover in the linear setting, the referred condition is not only sufficient but also necessary for the metric regularity of $\mathcal{G}^*$ at $\overline{x}$ for $(\bar{c},\bar{b})$.

The metric regularity is a basic quantitative property of mappings in variational analysis which is widely used in both theoretical and computational studies. In order to illustrate how this concept works in our context, let $\overline{x}$ be an optimal solution of $P(\bar{c},\bar{b})$ and let (c_a,b_a) and x_a be close enough approximations to $(\bar{c},\bar{b})$ and $\overline{x}$, respectively. Then problem $P(c_a,b_a)$ has an optimal solution whose distance to x_a is bounded by κ times $d((c_a,b_a),\mathcal{G}^*(x_a))$. The latter distance is usually easy to compute or estimate, while finding an exact solution of $P(c_a, b_a)$ might be considerably difficult. For instance, a possible choice of parameters which make x_a optimal is $c = \overline{c}$ and b such that x_a is feasible for $\sigma(b)$ and some suitably chosen constraints are active at x_a (according to the KKT condition). See section 3 for details. The metric regularity of a set-valued mapping turns out to be equivalent to the pseudo-Lipschitz property, also called the Aubin property, of the inverse mapping (see, for instance, [19], [25] and the references therein). Specifically, the Aubin property in our context reads as follows: There exist neighborhoods U of $\overline{x}$ and V of $(\overline{c}, \overline{b})$ and a constant $\kappa \geq 0$ such that

$$d\left(x^2, \mathcal{F}^*\left(c^1, b^1\right)\right) \leq \kappa d\left(\left(c^1, b^1\right), \left(c^2, b^2\right)\right) \tag{5}$$

for all $\left(c^1, b^1\right), \left(c^2, b^2\right) \in V$ and all $x^2 \in U \cap \mathcal{F}^*\left(c^2, b^2\right)$. Other Lipschitz/regularity properties also can be traced back to [19], [25].

In our context of problems (1), the metric regularity of $\mathcal{G}^*$ (i.e., the pseudo-Lipschitz property of $\mathcal{F}^*$) at a point of its graph is equivalent to the *strong Lipschitz stability* of $\mathcal{F}^*$ (see Lemma 5), which reads as follows: There exist open neighborhoods U of $\overline{x}$ and V of $(\overline{c}, \overline{b})$ and a constant $\kappa \geq 0$ such that $\mathcal{F}^*(c, b) \cap U$ is a singleton, $\{x(c,b)\}$, for all $(c, b) \in V$ and

$$\left\|x\left(c^1, b^1\right) - x\left(c^2, b^2\right)\right\| \leq \kappa \left\|\left(c^1, b^1\right) - \left(c^2, b^2\right)\right\| \text{ for all } \left(c^1, b^1\right), \left(c^2, b^2\right) \in V.$$

Note that because of the convexity of $\mathcal{F}^*(c, b)$, we already have $\mathcal{F}^*(c, b) = \{x(c, b)\}$ for all $(c, b) \in V$. In other words, the *strong Lipschitz stability* of $\mathcal{F}^*$ at $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ is equivalent to the local single-valuedness and Lipschitz continuity of $\mathcal{F}^*$ near $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$ [17], [19], [26]. The fact that the pseudo-Lipschitz property of the global optimal solution set mapping $\mathcal{S}$ of a parametric optimization problem implies strong Lipschitz stability of $\mathcal{S}$ holds for a rather general class of optimization problems (see again Lemma 5). In the particular case of linear problems, we can add as a third equivalent property the *local single-valuedness and continuity* of $\mathcal{F}^*$ (a Kojima-type stability condition under specific perturbations [21], [26]).

Section 5.3 in [20] clarifies the relationship between the strong Lipschitz stability and the strong Kojima stability. Specifically, as a straightforward consequence of Corollary 5.5 there, one obtains the equivalence between these two properties when applied to finite linear optimization problems. In this way, Theorem 16 below, confined to the linear case, extends the fulfillment of these equivalences to the case of infinitely many constraints.

That paper [20] was concerned with the strong Lipschitz stability of the stationary solution map (in the KKT sense) in our context of problems (1), with T finite, where the functions included in the model are assumed to belong to the class $C^{1,1}$, and under the general assumption of the Mangasarian–Fromowitz constraint qualification (MFCQ). The more general case in which the functions f and g also depend on a parameter $\tau \in \mathcal{T} \subset \mathbb{R}^r$ is dealt with in [19, sect. 8]. Note that if the constraint functions g_t of the convex semi-infinite problem (1) are differentiable, then the (extended) MFCQ is nothing else but the Slater CQ (i.e., the existence of a strict solution of the associated constraint system). The fulfillment of both the Slater condition and the boundedness (and nonemptiness) of the set of optimal solutions yields high stability for optimization problems in different frameworks (see, for instance, [18, Thm. 1] and [5, Thm. 4.2] in relation to the Lipschitz continuity of the optimal value).

There are different contributions to the stability theory for the feasible and the optimal set mappings in linear semi-infinite optimization. The article [10] analyzed the (Berge) lower semicontinuity of the feasible set mapping $\mathcal{F}$ in the more general context in which there is no continuity assumption and the parameters are $(a,b) \in (\mathbb{R}^n \times \mathbb{R})^T$, the latter being endowed with an appropriate extended distance. On the other hand, the lower and upper semicontinuity of $\mathcal{F}^*$ in the general context of parameters $(c,(a,b)) \in \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R})^T$ were analyzed in [5] in the linear case, and in [8] in the convex case. More details about stability of linear semi-infinite problems and their constraint systems in this general context (no continuity assumption) are gathered in [9, Chapters 6 and 10]. The *continuous case*, in which T is a compact Hausdorff space, the functions a and b are continuous on T, and all the parameters may be (continuously) perturbed, was analyzed, e.g., in [3] and [7]. Note also that classical parametric optimization (see, e.g., [1], [2], [16]) applies to this and more general settings by writing the constraints as one aggregated inequality, like $\max_{t\in T}(g_t(x) - b_t) \leq 0$ in the case of (1). In the current context of continuous perturbations of only the right-hand side of the system, the metric regularity of the mapping $\mathcal{G}$, in the linear case, was approached in [4].

Next, we summarize the structure of the paper. Section 2 gathers some preliminaries about convex analysis and multifunctions. Moreover we include here some results about the stability of $\mathcal{F}$ and its relation with continuity properties of $\mathcal{F}^*$. Specifically, Lemma 3 shows the equivalence among some relevant stability criteria concerning the feasible set. Proposition 4 provides a sufficient condition for the lower semicontinuity of $\mathcal{F}^*$, which constitutes a key step in the analysis of the metric regularity of $\mathcal{G}^*$. In section 3 we introduce, after some motivation, condition (10). Some consequences of this condition are gathered in Proposition 9. Theorem 10 shows that condition (10) is sufficient for the metric regularity of $\mathcal{G}^*$ in the convex case. Section 4 deals with the linear case. Theorem 16 establishes the equivalence between the specification of (10) for the linear case and several well-known stability concepts concerning the optimal set, including the metric regularity of $\mathcal{G}^*$. Finally, section 5 shows at a glance the main results of the paper.

2. Preliminaries and first results.

In this section we provide further notation and some preliminary results. Given $X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $conv\,(X)$ and $cone\,(X)$ the convex hull and the conical convex hull of X, respectively. We assume that $cone\,(X)$ always contains the zero vector of $\mathbb{R}^k$, 0_k. We shall also assume $conv\,(\emptyset) = \emptyset$ and $cone\,(\emptyset) := \{0_k\}$. If X is a closed convex set, $O^+(X)$ represents the recession cone of X.

If X is a subset of any topological space, $\operatorname{int}(X)$ and $\operatorname{cl}(X)$ will represent the interior and the closure of X, respectively. A typical element of $cone\,(\{x_i, i \in I\})$, where I is any index set, is represented as $\sum_{i\in I} \lambda_i x_i$, where $\lambda = (\lambda_i)_{i\in I}$ belongs to the cone $\mathbb{R}_+^{(I)}$ of all functions from I to $\mathbb{R}_+ := [0, +\infty[$ with finite support, i.e., taking positive values at only finitely many points of I. Generically, sequences will be indexed by $r \in \mathbb{N}$, and $\lim_r$ should be interpreted as $\lim_{r\to\infty}$.

Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper closed convex function. By $\partial h\,(x)$ we denote the *subdifferential* of h at x, and by $h0^+$ the *recession function* of h, i.e., the sublinear function whose epigraph is the recession cone of the epigraph of h.

Observe that our problem $P(c,b)$ is equivalent to the unconstrained problem

$$\text{(6)} \qquad \operatorname*{Inf}_{x\in\mathbb{R}^n} \left\{ h(x) := f(x) + c'x + \delta_{\mathcal{F}(b)}(x) \right\},$$

where $\delta_{\mathcal{F}(b)}$ is the indicator function of $\mathcal{F}(b)$ (i.e., $\delta_{\mathcal{F}(b)}(x) = 0$ if $x \in \mathcal{F}(b)$, and $\delta_{\mathcal{F}(b)}(x) = +\infty$ if $x \notin \mathcal{F}(b)$). We shall use the recession function of h, which, thanks to [24, Thm. 9.3], turns out to be

$$h0^+(y) = f0^+(y) + c'y + \delta_{\mathcal{F}(b)}0^+(y)$$
$$= f0^+(y) + c'y + \delta_{O^+(\mathcal{F}(b))}(y).$$

Associated with problem (1), for each $x \in \mathcal{F}(b)$ we consider

$$T_b(x) = \{t \in T \mid g_t(x) = b_t\} \quad \text{and} \quad A_b(x) = cone\left(\bigcup_{t \in T_b(x)} (-\partial g_t(x))\right).$$

Recall that $A_b(x) = \{0_n\}$ if $T_b(x) = \emptyset$. For our model (1), $\sigma(b)$ satisfies the *Slater condition* if $T_b(x^0)$ is empty for some $x^0 \in \mathcal{F}(b)$, in which case x^0 is referred to as a Slater point of $\sigma(b)$ (see [9, sect. 7.5]). Note that the continuity of $t \mapsto g_t(x^0)$ together with the compactness of T entails that x^0 is a Slater point of $\sigma(b)$ if and only if there exists some slack $\rho > 0$ such that $g_t(x^0) \leq b_t - \rho$ for all $t \in T$.

LEMMA 1. *Let $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ and $x \in \mathbb{R}^n$. One has the following for the parametric problem* (1):

(i) *If $\sigma(b)$ satisfies the Slater condition, then $A_b(x)$ is closed* [9, Thm. 7.9].

(ii) *KKT conditions* (see [9, (7.9) and Thm. 7.8]): *If $x \in \mathcal{F}(b)$ and $(c + \partial f(x)) \cap A_b(x) \neq \emptyset$, then $x \in \mathcal{F}^*(c, b)$. The converse holds when $\sigma(b)$ satisfies the Slater condition.*

Next we recall some well-known continuity concepts for set-valued mappings. If $\mathcal{Y}$ and $\mathcal{Z}$ are two metric spaces and $\mathcal{H} : \mathcal{Y} \rightrightarrows \mathcal{Z}$ is a set-valued mapping, $\mathcal{H}$ is said to be *lower semicontinuous* (lsc, in brief), in the classical sense of Berge, at $y \in \mathcal{Y}$ if, for each open set $W \subset \mathcal{Z}$ such that $W \cap \mathcal{H}(y) \neq \emptyset$, there exists an open set $U \subset \mathcal{Y}$, containing y, such that $W \cap \mathcal{H}(y^1) \neq \emptyset$ for each $y^1 \in U$. The mapping $\mathcal{H}$ is *upper semicontinuous* (usc, for short), in the sense of Berge, at $y \in \mathcal{Y}$ if, for each open set $W \subset \mathcal{Z}$ such that $\mathcal{H}(y) \subset W$, there exists an open neighborhood of y in $\mathcal{Y}$, U, such that $\mathcal{H}(y^1) \subset W$ for every $y^1 \in U$. We say that $\mathcal{H}$ is *closed* at $y \in \mathcal{Y}$ if for all sequences $\{y^r\} \subset \mathcal{Y}$ and $\{z^r\} \subset \mathcal{Z}$ satisfying $\lim_{r\to\infty} y^r = y$, $\lim_{r\to\infty} z^r = z$, and $z^r \in \mathcal{H}(y^r)$, one has $z \in \mathcal{H}(y)$. Obviously, $\mathcal{H}$ is closed on $\mathcal{Y}$ (at every point $y \in \mathcal{Y}$) if the graph of $\mathcal{H}$, $gph(\mathcal{H}) := \{(y, z) \in \mathcal{Y} \times \mathcal{Z} : z \in \mathcal{H}(y)\}$, is closed (in the product topology). In what follows, $rge(\mathcal{H})$ will represent the image set of $\mathcal{H}$.

The following property of our optimal set mapping $\mathcal{F}^*$ is a straightforward consequence of [1, Thm. 4.3.3] and will be used later on.

LEMMA 2. *Let $(\bar{c}, \bar{b}) \in \mathbb{R}^n \times C(T, \mathbb{R})$. Assume that $\mathcal{F}$ is lsc at $\bar{b}$ and $\mathcal{F}^*(\bar{c}, \bar{b})$ is nonempty and bounded. Then $\mathcal{F}^*$ is usc at $(\bar{c}, \bar{b})$.*

Note that our mapping $\mathcal{F}$ is closed on $C(T, \mathbb{R})$ due to the continuity of each g_t. The lower semicontinuity of $\mathcal{F}$ turns out to be equivalent to other stability properties referred above (see [12] for a discussion about conditions (i)–(iii) in the following lemma).

LEMMA 3. (See [4, Thm. 2.1] for the linear case with equality/inequality constraints.) *Let $\bar{b} \in rge(\mathcal{G})$. The following statements are equivalent:*

(i) *$\sigma(\bar{b})$ satisfies the Slater condition.*

(ii) *$\mathcal{F}$ is lsc at $\bar{b}$.*

(iii) $\bar{b} \in \operatorname{int}(rge(\mathcal{G}))$.

(iv) *$\mathcal{G}$ is metrically regular at any $x \in \mathcal{F}(\bar{b})$ for $\bar{b}$.*

(v)

$$0_n \notin conv\left(\bigcup_{t\in T_b(x)} \partial g_t(x)\right) \text{ for all } x\in\mathcal{F}\left(\overline{b}\right) \text{ such that } T_{\overline{b}}(x)\neq\emptyset. \tag{7}$$

Proof. (i)⇒(ii). Define the function $G:\mathbb{R}^n\times C(T,\mathbb{R})\to\mathbb{R}$ by

$$G(x,b) := \max_{t\in T}\left(g_t(x)-b_t\right).$$

Hence $\mathcal{F}(b)=\{x\,|\,G(x,b)\leq 0\}$. By classical parametric optimization (cf., e.g., [2], [16]), G is continuous, since $(t,x,b)\mapsto g_t(x)-b_t$ is continuous and T is nonempty and compact. Obviously, for given b, $G(\cdot,b)$ is convex. Since for $x\in\mathcal{F}\left(\overline{b}\right)$ we have $G(x,\overline{b})=0$ if and only if $T_{\overline{b}}(x)\neq\emptyset$, statement (i) is equivalent to the existence of $\overline{x}$ such that $G(\overline{x},\overline{b})<0$. Now, Theorem 12 in [16] applies.

(ii)⇒(iii). It comes straightforwardly from the definitions, taking into account that (iii) may be interpreted as $\sigma\left(b^1\right)$ being consistent ($\mathcal{F}\left(b^1\right)\neq\emptyset$) for all b^1 in some neighborhood of $\overline{b}$.

(iii)⇒(i). It follows from the following fact: For $\varepsilon>0$ small enough, $\mathcal{F}\left(b^\varepsilon\right)\neq\emptyset$, where $b^\varepsilon\in C\left(T,\mathbb{R}\right)$ is given by $b_t^\varepsilon := \overline{b}_t-\varepsilon$, $t\in T$. In this case, any feasible point of $\sigma\left(b^\varepsilon\right)$ is a Slater point of $\sigma\left(\overline{b}\right)$ with slack ε.

(iii)⇔(iv). This equivalence is established via the Robinson–Ursescu theorem (see, for instance, [6]) for mappings between Banach spaces having a closed convex graph. We have already mentioned that $gph(\mathcal{G})$ is closed, and it is also convex, due to the convexity of each g_t.

(i)⇔(v). With G as above, let $g(x):=G(x,\overline{b})$. Thus, (i) equivalently means that $g(\overline{x})<0$ is satisfied for some $\overline{x}$, which holds if and only if every point $x\in\mathcal{F}\left(\overline{b}\right)$ such that $T_{\overline{b}}(x)\neq\emptyset$ is not a minimum of g. By [15, Thm. VI.4.4.2], the latter is equivalent to the following fact: For every point $x\in\mathcal{F}\left(\overline{b}\right)$ such that $T_{\overline{b}}(x)\neq\emptyset$ we have

$$0_n\notin\partial g(x)=conv\left(\bigcup_{t\in T_b(x)}\partial g_t(x)\right),$$

and this is precisely (v). □

The following proposition accounts for some properties of $\mathcal{F}^*$ in relation to $\mathcal{F}$ (see also Lemma 2).

Proposition 4. (i) *If* $(\overline{c},\overline{b})\in\operatorname{int}(\Pi_s)$*, then* $\mathcal{F}^*(\overline{c},\overline{b})$ *is a nonempty bounded set.*

(ii) *Assume that* $(\overline{c},\overline{b})\in\operatorname{int}(\Pi_c)$ *and that* $\mathcal{F}^*(\overline{c},\overline{b})$ *is a nonempty bounded set. Then* $(\overline{c},\overline{b})\in\operatorname{int}(\Pi_s)$ *and* $\mathcal{F}^*(c,b)$ *is also a nonempty bounded set for* (c,b) *in a certain neighborhood of* $(\overline{c},\overline{b})$.

(iii) *If* $\mathcal{F}$ *is lsc at* $\overline{b}$*, then* $\mathcal{F}^*$ *is closed at* $\left(\overline{c},\overline{b}\right)$.

(iv) *If* $\mathcal{F}$ *is lsc at* $\overline{b}$ *and* $\mathcal{F}^*\left(\overline{c},\overline{b}\right)$ *is a singleton, then* $\mathcal{F}^*$ *is lsc at* $\left(\overline{c},\overline{b}\right)$.

Proof. (i) Let $(\overline{c},\overline{b})\in\operatorname{int}(\Pi_s)$, and assume that $\mathcal{F}^*(\overline{c},\overline{b})$ is unbounded. Take $u\in O^+(\mathcal{F}^*(\overline{c},\overline{b}))$, $u'u=1$, and consider the sequence in Π_c, $\left(\overline{c}-\frac{1}{r}u,\overline{b}\right)$, $r=1,2,\ldots$, which obviously converges to $(\overline{c},\overline{b})$. Now, for $\lambda\geq 0$ and $\overline{x}\in\mathcal{F}^*(\overline{c},\overline{b})\subset\mathcal{F}(\overline{b})$, and representing by v the optimal value of $P(\overline{c},\overline{b})$, we have

$$f\left(\overline{x}+\lambda u\right)+\left(\overline{c}-\frac{1}{r}u\right)'\left(\overline{x}+\lambda u\right)=v-\frac{1}{r}u'\overline{x}-\frac{\lambda}{r}.$$

By letting $\lambda \to +\infty$, it follows that the objective function of $P\left(\overline{c} - \frac{1}{r}u, \overline{b}\right)$ is unbounded from below, and this contradicts the assumption $(\overline{c}, \overline{b}) \in \operatorname{int}(\Pi_s)$.

(ii) Since $\mathcal{F}^*(\overline{c}, \overline{b})$ is nonempty and bounded, [15, Prop. IV.3.2.5] yields $\overline{h}0^+(y) > 0$ for all $y \neq 0_n$, where $\overline{h}$ is the function introduced in (6), associated to the nominal parameter $(\overline{c}, \overline{b})$. Since $\overline{h}0^+$ is lsc

$$\varepsilon := \min\{\overline{h}0^+(y) \mid \|y\|_* = 1\} > 0.$$

Consider any parameter (c, b) such that $\|c - \overline{c}\| < \varepsilon$ and that is close enough to $(\overline{c}, \overline{b})$ to be sure that $(c, b) \in \Pi_c$. If h is the associated function (see (6)) and $\|y\|_* = 1$, we can write

$$\begin{aligned} h0^+(y) &= f0^+(y) + c'y + \delta_{O^+(\mathcal{F}(b))}(y) \\ &= f0^+(y) + \overline{c}'y + \delta_{O^+(\mathcal{F}(\overline{b}))}(y) + (c - \overline{c})'y \\ &= \overline{h}0^+(y) + (c - \overline{c})'y \\ &\geq \overline{h}0^+(y) - \|c - \overline{c}\| \\ &> \overline{h}0^+(y) - \varepsilon \geq 0. \end{aligned} \tag{8}$$

Since (8) entails $h0^+(y) > 0$ for all $y \neq 0_n$, [15, Prop. IV.3.2.5] implies that $\mathcal{F}^*(c, b)$ is a nonempty bounded set.

(iii) Since $\mathcal{F}$ is closed at $\overline{b}$, this is a classical result; see [16, Thm. 8].

(iv) Since $\mathcal{F}^*\left(\overline{c}, \overline{b}\right)$ is a singleton, it holds by definition that $\mathcal{F}^*$ is lsc at $\left(\overline{c}, \overline{b}\right)$ if $\mathcal{F}^*$ is both usc at $\left(\overline{c}, \overline{b}\right)$ and nonempty-valued near $\left(\overline{c}, \overline{b}\right)$. The first property follows from Lemma 2, the second one from Corollary 9.1 in [16]. ∎

Problem (1) fits into the more general class of parametric problems given by

$$\begin{aligned} \mathcal{P}(c, b): \quad & \operatorname{Inf} f(x) + c'x \\ & \text{s.t. } x \in \mathcal{M}(b), \end{aligned}$$

where f is any real-valued function defined on $\mathbb{R}^n$, $\mathcal{M}$ is any multifunction which maps a metric space $\mathcal{Y}$ to $\mathbb{R}^n$, and $(c, b) \in \mathbb{R}^n \times \mathcal{Y}$ varies in some neighborhood of $(\overline{c}, \overline{b}) \in \mathbb{R}^n \times \mathcal{Y}$. If we define

$$\mathcal{F}^*(c, b) := \arg\min\{f(x) + c'x \mid x \in \mathcal{M}(b)\},$$

we obtain the following result without any assumption about continuity.

LEMMA 5 (Corollary 4.7 in [19]). *Let $((\overline{c}, \overline{b}), \overline{x}) \in gph(\mathcal{F}^*)$. Then $\mathcal{F}^*$ is pseudo-Lipschitz at $((\overline{c}, \overline{b}), \overline{x})$ if and only if $\mathcal{F}^*$ is strongly Lipschitz stable at this point.*

Proof. To show the nontrivial direction, let $\mathcal{F}^*$ be pseudo-Lipschitz at $((\overline{c}, \overline{b}), \overline{x})$. Hence, by Corollary 4.7 in [19], $\mathcal{F}^*(c, b)$ is a singleton for (c, b) near $(\overline{c}, \overline{b})$. This implies strong Lipschitz stability at (and hence, by definition of that stability, *near*) $((\overline{c}, \overline{b}), \overline{x})$. ∎

3. A sufficient condition for the metric regularity of $\mathcal{G}^*$. This section provides a KKT-type condition which is sufficient for the metric regularity $\mathcal{G}^*$ at $\overline{x}$ for $\left(\overline{c}, \overline{b}\right) \in \mathcal{G}^*(\overline{x})$ in the context of convex problems (1). The relationship between this condition and the strong uniqueness of optimal solutions is explored, too. The specification of this KKT-type property for linear problems (4) turns out to be also necessary for the metric regularity. The next example partially motivates this algebraic condition in the linear case.

Example 6. Consider the problem, in $\mathbb{R}^2$ (with the Euclidean norm),

$$P\left(\bar{c},\bar{b}\right) := \operatorname{Inf}\left\{x_1 \mid x_1 - x_2 \geq 0,\ x_1 + x_2 \geq 0,\ x_1 \geq 0\right\}.$$

Here $\bar{c} = (1,0)'$ and $\bar{b} = 0_3$.

One has $\mathcal{F}^*\left(\bar{c},\bar{b}\right) = \{0_2\}$. If we consider the perturbed problem $P\left(c^r, b^r\right)$, with $b^r := (0,0,1/r)'$ and $c^r = \left(1, -1/r^2\right)$, we have $\mathcal{F}^*\left(c^r, b^r\right) = \left\{\left(\frac{1}{r}, \frac{1}{r}\right)\right\}$. So, by taking $x^r = \left(\frac{1}{r}, 0\right)'$, we obtain

$$d\left(x^r, \mathcal{F}^*\left(c^r, b^r\right)\right) = \frac{1}{r} \quad \text{and} \quad d\left(\left(c^r, b^r\right), \mathcal{G}^*\left(x^r\right)\right) \leq d\left(\left(c^r, b^r\right), \left(\bar{c}, b^r\right)\right) = \frac{1}{r^2}.$$

Hence, $d\left(x^r, \mathcal{F}^*\left(c^r, b^r\right)\right) \geq r d\left(\left(c^r, b^r\right), \mathcal{G}^*\left(x^r\right)\right)$, $r = 1, 2, \ldots$. Therefore, $\mathcal{G}^*$ is not metrically regular at 0_2 for $\left(\bar{c},\bar{b}\right)$.

The key fact in this example is that $\bar{c}$ belongs to the convex cone generated by *one* vector, associated with the active constraints in $\overline{x}$, in the *two*-dimensional Euclidean space. The following property, referred to as a given $\left(\overline{x}, \left(\bar{c},\bar{b}\right)\right) \in gph(\mathcal{G}^*)$ in the linear case (4), avoids the previous situation (here $|D|$ denotes the cardinality of D):

$$\text{(9)} \qquad \begin{aligned} &\sigma\left(\bar{b}\right) \text{ satisfies the Slater condition and there is no } D \subset T_{\bar{b}}\left(\overline{x}\right) \\ &\text{with } |D| < n \text{ such that } \bar{c} \in cone\left(\{a_t,\ t \in D\}\right). \end{aligned}$$

The following natural extension of (9) for the convex problem (1) will play a crucial role in this section; in fact, it constitutes the announced sufficient condition for the metric regularity of $\mathcal{G}^*$ at $\left(\overline{x}, \left(\bar{c},\bar{b}\right)\right)$:

$$\text{(10)} \qquad \begin{aligned} &\sigma\left(\bar{b}\right) \text{ satisfies the Slater condition and there is no } D \subset T_{\bar{b}}\left(\overline{x}\right) \\ &\text{with } |D| < n \text{ such that } \left(\bar{c} + \partial f(\overline{x})\right) \cap cone\left(\bigcup_{t \in D}\left(-\partial g_t\left(\overline{x}\right)\right)\right) \neq \emptyset. \end{aligned}$$

Remark 7. Observe that condition (9) does not imply the linear independence of $\{a_t,\ t \in T_{\bar{b}}(\overline{x})\}$. Consider the example resulting from replacing the third constraint in Example 6 with any of the other two (which would appear twice in the system).

Remark 8. In the case $n = 1$, condition (10) reads as follows: $\sigma\left(\bar{b}\right)$ satisfies the Slater condition and $0 \notin \bar{c} + \partial f(\overline{x})$ (which entails $T_{\bar{b}}\left(\overline{x}\right) \neq \emptyset$).

PROPOSITION 9. *Assume that* $\left(\overline{x}, \left(\bar{c},\bar{b}\right)\right) \in gph(\mathcal{G}^*)$ *verifies* (10). *Then the following conditions hold:*

(i) *There exists a neighborhood* W *of* $\left(\overline{x}, \left(\bar{c},\bar{b}\right)\right)$ *such that* (10) *is satisfied when* $\left(\overline{x}, \left(\bar{c},\bar{b}\right)\right)$ *is replaced by any* $(x,(c,b)) \in W \cap gph(\mathcal{G}^*)$.

(ii) *There exist* $u \in \partial f(\overline{x})$ *as well as some* $u_{t_i} \in -\partial g_{t_i}\left(\overline{x}\right)$, $t_i \in T_{\bar{b}}\left(\overline{x}\right)$, *and some* $\lambda_i > 0$ *for* $i \in \{1,\ldots,n\}$, *such that* $\{u_{t_1},\ldots,u_{t_n}\}$ *is a basis of* $\mathbb{R}^n$ *and*

$$u + \bar{c} = \sum_{i=1}^{n} \lambda_i u_{t_i}.$$

(iii) $\mathcal{F}^*\left(\bar{c},\bar{b}\right) = \{\overline{x}\}$.

(iv) $\mathcal{F}^*$ *is lsc at* $\left(\bar{c},\bar{b}\right)$.

As a consequence of the previous statements, one has the following condition:

(v) *There exists a neighborhood V of $(\bar{c}, \bar{b})$ such that $\mathcal{F}^*$ is single-valued and continuous on V.*

Proof. (i) From the equivalence (i)⇔(iii) in Lemma 3, it is clear that $\sigma(b)$ fulfills the Slater condition for b close enough to $\bar{b}$. Now, reasoning by contradiction, assume that there exists $\{(x^r, (c^r, b^r))\} \subset gph(\mathcal{G}^*)$ converging to $(\overline{x}, (\bar{c}, \bar{b}))$ as well as some subgradients $u^r \in \partial f(x^r)$, $u^r_{t^r_i} \in -\partial g_{t^r_i}(x^r)$, $t^r_i \in T_{b^r}(x^r)$, $\lambda^r_i \geq 0$, $i = 1, \ldots, n-1$, $r = 1, 2, \ldots$, such that we can write

$$u^r + c^r = \sum_{i=1}^{n-1} \lambda^r_i u^r_{t^r_i}. \tag{11}$$

In this expression we have made use of the convexity of the involved subdifferential sets.

For each $i \in \{1, \ldots, n-1\}$ the sequence $\{t^r_i\}$ has a subsequence (still denoted by $\{t^r_i\}$, for simplicity) converging to certain $\bar{t}_i \in T_{\bar{b}}(\overline{x})$, since T is compact and $g_{\bar{t}_i}(\overline{x}) - \bar{b}_{\bar{t}_i} = \lim_r \left(g_{t^r_i}(x^r) - b^r_{t^r_i}\right) = 0$. Let us see that the sequence $\{\gamma_r\}_{r\in\mathbb{N}}$ given by $\gamma_r := \sum_{i=1}^{n-1} \lambda^r_i$, $r = 1, 2, \ldots$, must be bounded. Otherwise, we may assume without loss of generality (considering suitable subsequences) that $\lim_{r\to\infty} \gamma_r = +\infty$ and the sequence $\left\{\frac{\lambda^r_i}{\gamma_r}\right\}_{r\in\mathbb{N}}$ converges to certain $\mu_i \geq 0$ for each $i \in \{1, \ldots, n-1\}$. So, dividing by γ_r in (11) and letting $r \to +\infty$ we have (considering again appropriate subsequences of $\left\{u^r_{t^r_i}\right\}_{r\in\mathbb{N}}$ for each i)

$$\begin{aligned} &0_n = \sum_{i=1}^{n-1} \mu_i u_{\bar{t}_i}, \\ &\text{with } \sum_{i=1}^{n-1} \mu_i = 1 \ \text{ and } \ u_{\bar{t}_i} := \lim\nolimits_r u^r_{t^r_i} \in -\partial g_{\bar{t}_i}(\overline{x}), \quad i = 1, \ldots, n-1, \end{aligned} \tag{12}$$

where we have applied [24, Thm. 24.5] to sequences $\left\{g_{t^r_i}\right\}_{r\in\mathbb{N}}$, $i = 1, \ldots, n-1$, and $\{x^r\}_{r\in\mathbb{N}}$ (here the continuity of $t \mapsto g_t(x)$, for all $x \in \mathbb{R}^n$, is essential to allow the use of the referred theorem). In this way we attain a contradiction with (7) in Lemma 3.

Once we have established the boundedness of $\{\gamma_r\}_{r\in\mathbb{N}}$, we may assume without loss of generality that, for each $i \in \{1, \ldots, n-1\}$, the sequence $\{\lambda^r_i\}_{r\in\mathbb{N}}$ converges to certain $\beta_i \geq 0$, $\left\{u^r_{t^r_i}\right\}_{r\in\mathbb{N}}$ converges again to certain $u_{\bar{t}_i} \in -\partial g_{\bar{t}_i}(\overline{x})$, and $\{u^r\}_{r\in\mathbb{N}}$ converges to some $u \in \partial f(\overline{x})$ (appealing again to [24, Thm. 24.5]). Thus, letting $r \to \infty$ in (11) we obtain

$$u + \bar{c} = \sum_{i=1}^{n-1} \beta_i u_{\bar{t}_i}, \text{ with } \left\{\bar{t}_1, \ldots, \bar{t}_{n-1}\right\} \subset T_{\bar{b}}(\overline{x}),$$

contradicting (10).

(ii) It follows easily from the KKT conditions (see Lemma 1), property (10), and Carathéodory's theorem.

(iii) Let $u + \bar{c}$ be represented as in (ii). If there exists $y \in \mathcal{F}^*(\bar{c}, \bar{b}) \setminus \{\overline{x}\}$, then we have, by using convexity of f and taking into account

$$0 \geq g_{t_i}(y) - \bar{b}_{t_i} = g_{t_i}(y) - g_{t_i}(\overline{x}) \geq -u'_{t_i}(y - \overline{x})$$

as well as $\lambda_i > 0$, $i = 1, 2, \ldots, n$,

$$0 = f(y) + \bar{c}'y - f(\overline{x}) - \bar{c}'\overline{x} \geq (u + \bar{c})'(y - \overline{x})$$

$$= \sum_{i=1}^{n} \lambda_i u_{t_i}'(y - \overline{x}) \geq 0.$$

Thus, we obtain $u_{t_i}'(y - \overline{x}) = 0$ for $i = 1, \ldots, n$, contradicting the fact that $\{u_{t_1,\ldots,}u_{t_n}\}$ is a basis of $\mathbb{R}^n$.

(iv) It is a straightforward consequence of (iii) above and Proposition 4(iv) (recall also that (i)⇔(ii) in Lemma 3).

(v) Take a neighborhood $U_0 \times V_0$ of $\left(\overline{x}, \left(\bar{c}, \bar{b}\right)\right)$ contained in certain W verifying (i). Due to (iv) we may consider a neighborhood of $\left(\bar{c}, \bar{b}\right)$, say $V \subset V_0$, such that $\mathcal{F}^*(c, b) \cap U_0 \neq \emptyset$ for all $(c, b) \in V$. Now, for each $(c, b) \in V$, there exists $x \in \mathcal{F}^*(c, b) \cap U_0$, and so $(x, (c, b)) \in W \cap gph(\mathcal{G}^*)$ and (i) together with (iii) entail $\mathcal{F}^*(c, b) = \{x\}$. Finally, the continuity of the single-valued mapping $\mathcal{F}^*|_V$ comes from (i) and (iv). ∎

Next we present a sufficient condition for metric regularity of $\mathcal{G}^*$. By Lemma 5, the latter is equivalent to the strong Lipschitz stability of $\mathcal{F}^*$.

THEOREM 10. *For the convex semi-infinite program* (1)*, let* $\left(\overline{x}, \left(\bar{c}, \bar{b}\right)\right) \in gph(\mathcal{G}^*)$*. If condition* (10) *holds, then* $\mathcal{G}^*$ *is metrically regular at* $\overline{x}$ *for* $\left(\bar{c}, \bar{b}\right)$*.*

Proof. Reasoning by contradiction, assume that (10) holds, but $\mathcal{G}^*$ is not metrically regular at $\overline{x}$ for $\left(\bar{c}, \bar{b}\right)$. According to the equivalence between metric regularity of a mapping and the Aubin property of its inverse (see (5)), there must exist a sequence $\{x^r\}_{r\in\mathbb{N}} \subset \mathbb{R}^n$ converging to $\overline{x}$ and two sequences of parameters $\{(c^r, b^r)\}_{r\in\mathbb{N}}$ and $\left\{\left(\bar{c}^r, \bar{b}^r\right)\right\}_{r\in\mathbb{N}}$, both converging to $\left(\bar{c}, \bar{b}\right)$, such that, for all $r \in \mathbb{N}$, $x^r \in \mathcal{F}^*(c^r, b^r)$ and

$$d\left(x^r, \mathcal{F}^*\left(\bar{c}^r, \bar{b}^r\right)\right) > rd\left((c^r, b^r), \left(\bar{c}^r, \bar{b}^r\right)\right). \tag{13}$$

Because of condition (v) in Proposition 9 we may assume without loss of generality that, for all r, $\mathcal{F}^*\left(\bar{c}^r, \bar{b}^r\right)$ is a singleton, say $\mathcal{F}^*\left(\bar{c}^r, \bar{b}^r\right) = \{\overline{x}^r\}$. The continuity of $\mathcal{F}^*$ at $\left(\bar{c}, \bar{b}\right)$ ensures that the sequence $\{\overline{x}^r\}$ converges to $\overline{x}$ (see again Proposition 9(v)). Moreover (13) ensures, for all r, $x^r \neq \overline{x}^r$ and

$$\frac{\sup_{t\in T}\left|b_t^r - \bar{b}_t^r\right|}{\|x^r - \overline{x}^r\|} < \frac{1}{r}. \tag{14}$$

According to conditions (i) and (ii) in Proposition 9 we can write, for r large enough,

$$u^r + c^r = \sum_{i=1}^{n} \lambda_i^r u_{t_i^r}^r \qquad \text{and} \qquad \overline{u}^r + \bar{c}^r = \sum_{i=1}^{n} \overline{\lambda}_i^r \overline{u}_{\bar{t}_i^r}^r \tag{15}$$

for certain subgradients $u^r \in \partial f(x^r)$, $\overline{u}^r \in \partial f(\overline{x}^r)$, $u_{t_i^r}^r \in -\partial g_{t_i^r}(x^r)$, $\overline{u}_{\bar{t}_i^r}^r \in -\partial g_{\bar{t}_i^r}(\overline{x}^r)$, associated with certain indices $t_i^r \in T_{b^r}(x^r)$ and $\bar{t}_i^r \in T_{\bar{b}^r}(\overline{x}^r)$, and certain positive scalars $\lambda_i^r, \overline{\lambda}_i^r$ for $i = 1, 2, \ldots, n$. Moreover, following the same argument as in the proof of Proposition 9(i), we may assume that for each $i = 1, \ldots, n$, the sequences $\left\{\lambda_i^r\right\}_{r\in\mathbb{N}}$ and $\left\{\overline{\lambda}_i^r\right\}_{r\in\mathbb{N}}$ converge to some λ_i and $\overline{\lambda}_i$, respectively. We may also assume that, for each i, the sequences $\left\{t_i^r\right\}_{r\in\mathbb{N}}$ and $\left\{\bar{t}_i^r\right\}_{r\in\mathbb{N}}$ involved in (15) converge to t_i and $\bar{t}_i$, respectively, both belonging to $T_{\bar{b}}(\overline{x})$, and that $\{u^r\}_{r\in\mathbb{N}}$, $\{\overline{u}^r\}_{r\in\mathbb{N}}$, $\left\{u_{t_i^r}^r\right\}_{r\in\mathbb{N}}$, and $\left\{\overline{u}^r_{t^r_i}\right\}_{r\in\mathbb{N}}$ converge to certain u, $\overline{u}\in\partial f(\overline{x})$, $u_{t_i}\in-\partial g_{t_i}(\overline{x})$, and $\overline{u}_{\bar{t}_i}\in-\partial g_{\bar{t}_i}(\overline{x})$, respectively. Thus (15) leads us to

$$u+\overline{c}=\sum_{i=1}^n\lambda_i u_{t_i}\quad\text{and}\quad\overline{u}+\overline{c}=\sum_{i=1}^n\overline{\lambda}_i\overline{u}_{\bar{t}_i}. \tag{16}$$

Moreover, condition (10) together with Carathéodory's theorem ensures all λ_i and $\overline{\lambda}_i$ are positive and that, at the same time, $\{u_{t_1},\dots,u_{t_n}\}$ and $\left\{\overline{u}_{\bar{t}_1},\dots,\overline{u}_{\bar{t}_n}\right\}$ are both bases of $\mathbb{R}^n$.

On the other hand, since, for each i and each r, we have $g_{t^r_i}(x^r)=b^r_{t^r_i}$, and $g_{t^r_i}(\overline{x}^r)\leq\overline{b}^r_{t^r_i}$ (recall $t^r_i\in T_{b^r}(x^r)$ and $\overline{x}^r\in\mathcal{F}(\overline{b}^r)$), we can write

$$u'_{t^r_i}\frac{x^r-\overline{x}^r}{\|x^r-\overline{x}^r\|}=-u'_{t^r_i}\frac{\overline{x}^r-x^r}{\|x^r-\overline{x}^r\|}\leq\frac{g_{t^r_i}(\overline{x}^r)-g_{t^r_i}(x^r)}{\|x^r-\overline{x}^r\|}\leq\frac{\overline{b}^r_{t^r_i}-b^r_{t^r_i}}{\|x^r-\overline{x}^r\|}<\frac{1}{r}, \tag{17}$$

where the last inequality comes from (14). By considering again a suitable subsequence, it is clear that $\left\{\frac{x^r-\overline{x}^r}{\|x^r-\overline{x}^r\|}\right\}_{r\in\mathbb{N}}$ may be assumed to converge to some $z\in\mathbb{R}^n$ with $\|z\|=1$. Hence letting $r\to\infty$ in (17) we obtain $u'_{t_i}z\leq 0$ for all $i=1,\dots,n$. Consequently, (16) ensures

$$(u+\overline{c})'z\leq 0. \tag{18}$$

A completely symmetric argument entails $\overline{u}'_{\bar{t}_i}z\geq 0$ for $i=1,\dots,n$, and, hence,

$$(\overline{u}+\overline{c})'z\geq 0. \tag{19}$$

This yields $u'z\leq\overline{u}'z$. To show that we have even equality, we note that by convexity of f,

$$f(x^r)\geq f(\overline{x}^r)+(\overline{u}^r)'(x^r-\overline{x}^r)\quad\text{and}\quad f(\overline{x}^r)\geq f(x^r)+(u^r)'(\overline{x}^r-x^r).$$

This implies

$$(\overline{u}^r)'(x^r-\bar{x}^r)\leq f(x^r)-f(\overline{x}^r)\leq(u^r)'(x^r-\bar{x}^r).$$

Hence, dividing by $\|x^r-\bar{x}^r\|$ and taking the limit yields $\overline{u}'z\leq u'z$, which establishes $\overline{u}'z=u'z$. Consequently, expressions (18) and (19) coincide, and then

$$(u+\overline{c})'z=(\overline{u}+\overline{c})'z=0.$$

Finally, appealing to the first equality of (16), and recalling that $u'_{t_i}z\leq 0$ and $\lambda_i>0$ for all i, we conclude $u'_{t_i}z=0$ for $i=1,\dots,n$. This, recalling that $z\neq 0_n$, represents a contradiction with the fact that $\{u_{t_1},\dots,u_{t_n}\}$ is a basis of $\mathbb{R}^n$. This completes the proof. $\square$

Remark 11. Condition (10) is not necessary for metric regularity of the mapping $\mathcal{G}^*$. Just consider the optimization problem, in $\mathbb{R}^2$,

$$\begin{aligned}P(c,b):\ \ &\text{Inf } x_1^2+x_2+c_1x_1+c_2x_2\\ &\text{s.t. } x_1\geq b_1,\ x_2\geq b_2.\end{aligned}$$

Note that, in a neighborhood of $(\bar{c},\bar{b})=(0_2,0_2)$, $\mathcal{F}^*$ is the Lipschitz function given by $\mathcal{F}^*(c,b)=\{(\max\{-c_1/2,b_1\},b_2)\}$, and then $\mathcal{G}^*$ is metrically regular at $\bar{x}=0_2$ for $(\bar{c},\bar{b})$. However, condition (10) fails.

Remark 12. In fact, condition (10) is in general rather strong for metric regularity: as we will see, it implies a first order growth condition on f at $\overline{x}$ with respect to $\sigma(\overline{b})$, namely, the strong uniqueness of $\overline{x}$ as a minimizer of $P\left(\overline{c},\overline{b}\right)$ (see (20)), and moreover at least n constraints have to be active at $\overline{x}$. It is well known for finite nonlinear optimization problems with twice differentiable data that already certain second order growth conditions—which also typically hold in the situation of less than n active constraints—are sufficient and necessary for metric regularity of $\mathcal{G}^*$; see, e.g., [19, Chap. 8] and [20]. Generalizing this to the nonlinear semi-infinite case remains an open problem. However, in the next section we will see that for linear semi-infinite programs, condition (10) is indeed needed for metric regularity of $\mathcal{G}^*$ at $\overline{x}$ for $(\overline{c},\overline{b})$.

The rest of this section is concerned with the relationship between condition (10) and the strong uniqueness of a minimizer in the context of convex optimization. For continuously differentiable data f and g_t and under the Slater condition, property (ii) of Proposition 9 (recall that it is a consequence of condition (10)) is known as a sufficient condition for $\overline{x}$ to be a (locally) strongly unique minimizer of $P(\overline{c},\overline{b})$; see Theorem 3.1.16 in [14]. In the linear case, condition (10) turns out to be equivalent even to persistence of strong unicity under small parameter changes (see section 4 for details). In the following paragraphs we show how condition (10) is still sufficient for the latter property but no longer necessary.

Here, we say that $x \in \mathcal{F}(b)$ is a *strongly unique minimizer* of $P(c,b)$ if there exists a positive scalar α such that

$$f(y)+c'y \geq f(x)+c'x+\alpha\left\|y-x\right\| \text{ for all } y \in \mathcal{F}(b). \tag{20}$$

Obviously, in that case $\mathcal{F}^*(c,b)=\{x\}$. (Note that the convexity assumptions allow us to formulate the previous definition in global terms, not only in a neighborhood of x.) The following lemma characterizes the strong uniqueness of optimal solutions in terms of perturbations of vector c (which generalizes the linear version given in [9, Thm. 10.5]).

LEMMA 13. *A point x is the* strongly unique optimal solution *of $P(c,b)$ if and only if there exists $\varepsilon>0$ such that $\|\widetilde{c}-c\|<\varepsilon$ implies $x\in\mathcal{F}^*(\widetilde{c},b)$ (in fact, for possibly smaller ε, x is the strongly unique solution of $P(\widetilde{c},b)$).*

Proof. According to [23, Chap. 5, Lem. 3] and [24, Thm. 23.8], x is a strongly unique optimal solution of $P(c,b)$ or, equivalently, of the problem

$$\operatorname*{Inf}_{x\in\mathbb{R}^n}\left\{f(x)+c'x+\delta_{\mathcal{F}(b)}(x)\right\},$$

if and only if

$$0_n\in\operatorname{int}\{c+\partial(f+\delta_{\mathcal{F}(b)})(x)\}=c+\operatorname{int}\{\partial(f+\delta_{\mathcal{F}(b)})(x)\}$$

holds. The latter is equivalent to

$$0_n\in\widetilde{c}+\partial(f+\delta_{\mathcal{F}(b)})(x) \text{ for } \widetilde{c} \text{ close enough to } c,$$

i.e., $x\in\mathcal{F}^*(\widetilde{c},b)$ for $\widetilde{c}$ close enough to c. To ensure the last assertion, just take $\widetilde{c}$ such that $0_n\in\widetilde{c}+\operatorname{int}\left\{\partial(f+\delta_{\mathcal{F}(b)})(x)\right\}$. ∎

PROPOSITION 14. *If condition* (10) *holds at $\left(\overline{x},\left(\bar{c},\overline{b}\right)\right)\in gph(\mathcal{G}^*)$, then $\overline{x}$ is the strongly unique optimal solution of $P\left(\overline{c},\overline{b}\right)$.*

Proof. From Proposition 9(ii) there exist $u\in\partial f(\overline{x})$ as well as some $u_{t_i}\in-\partial g_{t_i}(\overline{x})$, $t_i\in T_{\overline{b}}(\overline{x})$, and some $\lambda_i>0$ for $i\in\{1,\ldots,n\}$, such that $\{u_{t_1},\ldots,u_{t_n}\}$ is a basis of $\mathbb{R}^n$ and

$$u + \bar{c} = \sum_{i=1}^{n} \lambda_i u_{t_i}.$$

So, $u + \bar{c} \in \operatorname{int}(cone\left(\{u_{t_1}, \ldots, u_{t_n}\}\right))$. Hence, if $\|\widetilde{c} - \bar{c}\|$ is small enough, then

$$u + \widetilde{c} \in cone\left(\{u_{t_1}, \ldots, u_{t_n}\}\right),$$

which entails $\overline{x} \in \mathcal{F}^*\left(\widetilde{c}, \overline{b}\right)$. Thus, applying the previous lemma, $\overline{x}$ is the strongly unique optimal solution of $P\left(\overline{c}, \overline{b}\right)$. $\square$

Remark 15. Actually, under condition (10), we have that $\left(\overline{c}, \overline{b}\right) \in \operatorname{int}(\{(c,b) :$ $P\left(c,b\right)$ has a strongly unique optimal solution$\})$ as a consequence of Proposition 9(i) and (v) (the latter ensures that all problems in a certain neighborhood have optimal solutions and (i) entails that these solutions are strongly unique). However, the converse statement does not hold. Just consider the parametrized convex problem, in which condition (10) fails trivially ($|T| = 1$, while the problem is posed in $\mathbb{R}^2$):

$$P\left(c,b\right) := \operatorname{Inf}\left\{c_1 x_1 + c_2 x_2 \mid |x_1| - x_2 \leq b\right\},$$

around $\left(\overline{c}, \overline{b}\right) = ((0,1)', 0)$. In fact, one can easily check that

$$\mathcal{F}^*\left(c,b\right) = \{(0,-b)\} \text{ if } \|c - \overline{c}\| < \frac{1}{\sqrt{2}},$$

and, since $\mathcal{F}^*\left(c,b\right)$ does not depend on c, we immediately conclude that $(0,-b)$ is a strongly unique optimal solution of $P\left(c,b\right)$ when $\|c - \overline{c}\| < \frac{1}{\sqrt{2}}$. (We used the Euclidean norm.)

Finally, note that the metric regularity property is sufficient neither for condition (9) nor for strong uniqueness. Just consider the example of Remark 11 and note that $\overline{x}$ is not a locally strongly unique minimizer of $P(0_2, 0_2)$, considering the feasible ray $\{(t,0) \mid t \geq 0\}$.

4. Characterization of the metric regularity of $\mathcal{G}^*$ for linear problems. The following theorem provides the announced characterizations of the metric regularity of $\mathcal{G}^*$ for linear semi-infinite problems (4). Note that condition (v) is nothing else but (9). Moreover, condition (vi) comes from adapting to our notation Nürnberger's condition introduced in [22]. Actually, [22, Thm. 1.4] provides the counterpart of the equivalence (vi)$\Leftrightarrow$(vii) in the context in which perturbations of the a_t's are also allowed. The equivalence also holds, requiring only the boundedness of the a_t's, without continuity assumptions in the model (see [13, Thm. 4.1]).

THEOREM 16. *For the linear semi-infinite program* (4)*, let* $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in gph(\mathcal{G}^*)$*. Then the following conditions are equivalent:*

(i) $\mathcal{G}^*$ *is metrically regular at* $\overline{x}$ *for* $\left(\overline{c}, \overline{b}\right)$*.*

(ii) $\mathcal{F}^*$ *is strongly Lipschitz stable at* $\left(\left(\overline{c}, \overline{b}\right), \overline{x}\right)$*.*

(iii) $\mathcal{F}^*$ *is locally single-valued and continuous in some neighborhood of* $(\overline{c}, \overline{b})$*.*

(iv) $\mathcal{F}^*$ *is single-valued in some neighborhood of* $\left(\overline{c}, \overline{b}\right)$*.*

(v) $\sigma\left(\overline{b}\right)$ *satisfies the Slater condition and there is no* $D \subset T_{\overline{b}}\left(\overline{x}\right)$ *with* $|D| < n$ *such that* $\overline{c} \in cone\left(\{a_t,\ t \in D\}\right)$*.*

(vi) $\sigma\left(\overline{b}\right)$ *satisfies the Slater condition and for each* $D \subset T_{\overline{b}}\left(\overline{x}\right)$ *with* $|D| = n$ *such that* $\overline{c} \in cone\left(\{a_t,\ t \in D\}\right)$*; we have that all the possible subsets with* n *elements of* $\{a_t,\ t \in D\} \cup \{\overline{c}\}$ *are linearly independent.*

(vii) $(\bar{c}, \bar{b}) \in \operatorname{int}(\{(c, b) : \mathcal{F}^*(c, b)$ *consists of a strongly unique minimizer*$\})$.

Proof. The equivalence (i)⇔(ii) is nothing else but Lemma 5.

(ii)⇒(iii)⇒(iv). They are obvious consequences of the respective definitions.

(iv)⇒(v). From (iv) we immediately conclude that $(\bar{c}, \bar{b}) \in \operatorname{int}(rge(\mathcal{G}^*))$, which obviously implies $\bar{b} \in \operatorname{int}(rge(\mathcal{G}))$ and, from Lemma 3, $\mathcal{G}$ is metrically regular at $\overline{x}$ for $\bar{b}$ and $\sigma(\bar{b})$ satisfies the Slater condition. In fact, if $S(\bar{b})$ denotes the set of Slater points of $\sigma(\bar{b})$, then one has $S(\bar{b}) = \operatorname{int}(\mathcal{F}(\bar{b}))$ [9, Ex. 6.1]. Take $\widehat{x} \in S(\bar{b})$ and define, for each $r \in \mathbb{N}$,

$$x^r := \overline{x} + \frac{1}{r}(\widehat{x} - \overline{x}) \in \operatorname{int}(\mathcal{F}(\bar{b}))$$

(by the accessibility lemma).

Suppose, reasoning by contradiction, that $\bar{c} = \sum_{i=1}^k \lambda_i a_{t_i}$, with $t_i \in T_{\bar{b}}(\overline{x})$ and $\lambda_i > 0$ for $i = 1, \ldots, k$ and $k < n$. Now choose $u \in \{a_{t_1}, \ldots, a_{t_k}\}^\perp$ with $\|u\| = 1$, whose existence is guaranteed by $k < n$. Then, since $x^r \in \operatorname{int}(\mathcal{F}(\bar{b}))$, there exists some scalar α_r such that $y^r := x^r + \alpha_r u \in \mathcal{F}(\bar{b})$, and we shall take $\alpha_r \in]0, 1/r]$. Define, for each $r \in \mathbb{N}$,

$$b_t^r := (1 - \varphi_r(t)) \min\{a_t' x^r, a_t' y^r\} + \varphi_r(t) \bar{b}_t,$$

where $\varphi_r : T \to [0, 1]$ is a continuous function verifying

$$\varphi_r(t) = 0 \text{ if } t \in \{t_1, \ldots, t_k\} \quad \text{ and } \quad \varphi_r(t) = 1 \text{ if } a_t' \overline{x} - \bar{b}_t \geq \tfrac{1}{r}.$$

The existence of such a φ_r is guaranteed by Urysohn's lemma. If $\left\{t \in T \mid a_t' \overline{x} - \bar{b}_t \geq \frac{1}{r}\right\}$ is empty, we take $\varphi_r \equiv 0$. Observe that $x^r, y^r \in \mathcal{F}(\bar{b})$ implies that $x^r, y^r \in \mathcal{F}(b^r)$ for all r. Moreover, from the choice of u, we have $\{t_1, \ldots, t_k\} \subset T_{b^r}(x^r) \cap T_{b^r}(y^r)$, and $\bar{c} = \sum_{i=1}^k \lambda_i a_{t_i}$ ensures $x^r, y^r \in \mathcal{F}^*(\bar{c}, b^r)$ for all r (see Lemma 1). Now, let us show that $\lim_{r\to\infty} b^r = \bar{b}$. In fact, in the nontrivial case $a_t' \overline{x} - \bar{b}_t < \frac{1}{r}$ (otherwise $b_t^r = \bar{b}_t$) we have

$$\begin{aligned}
\left|b_t^r - \bar{b}_t\right| &\leq (1 - \varphi(t)) \left|\min\{a_t' x^r, a_t' y^r\} - \bar{b}_t\right| \\
&\leq \max\left\{\left|a_t' x^r - \bar{b}_t\right|, \left|a_t' y^r - \bar{b}_t\right|\right\} \\
&\leq \left|a_t' x^r - \bar{b}_t\right| + \left|a_t'(y^r - x^r)\right| \\
&\leq \left|a_t'(x^r - \overline{x})\right| + \left(a_t' \overline{x} - \bar{b}_t\right) + \|a_t\|_* \frac{1}{r} \\
&\leq \frac{1}{r}\left(1 + (1 + \|\widehat{x} - \overline{x}\|) \max_{t\in T} \|a_t\|_*\right),
\end{aligned}$$

just recalling the definition of x^r. Hence

$$\left\|b^r - \bar{b}\right\|_\infty \leq \frac{1}{r}\left(1 + (1 + \|\widehat{x} - \overline{x}\|) \max_{t\in T} \|a_t\|_*\right).$$

In this way, we provided a sequence $\{b^r\}_{r\in\mathbb{N}}$ converging to $\bar{b}$ such that $\mathcal{F}^*(\bar{c}, b^r)$ is not a singleton, which contradicts (iv).

(v)⇒(i). This follows from Theorem 10.

(v)$\Leftrightarrow$(vi) comes from standard arguments of linear algebra. Once we have established the equivalence among all conditions (i) to (vi), note that (vi)$\Rightarrow$(vii) comes from [22, Thm. 1.4] by taking into account that perturbations (c, b) are a particular case of perturbations of all coefficients. Finally, (vii)$\Rightarrow$(iv) is trivial. $\square$

Remark 17. Example 4.6 in [20] shows that in the convex case (even for finite programs) the metric regularity of $\mathcal{G}^*$ (or, equivalently, strong Lipschitz stability of $\mathcal{F}^*$) does not necessarily hold if $\mathcal{F}^*$ is Kojima-stable (locally single-valued and continuous). This is in contrast to the linear semi-infinite case treated in the foregoing theorem.

5. Concluding remarks.

The following diagram summarizes the main results of the paper concerning the convex case (1). The question of whether or not the strong uniqueness of an optimal solution for (c, b) near $(\bar{c}, \bar{b})$ implies the metric regularity of $\mathcal{G}^*$ at $(\overline{x}, (\bar{c}, \bar{b}))$ remains an open problem. Observe that condition (10) strictly implies the others in the diagram. Nevertheless, it is the only one which can be checked from the nominal problem's data, without involving parameters in a neighborhood.

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{Condition (10)}\\ \text{at } (\overline{x},(\bar{c},\bar{b}))\in gph\mathcal{G}^*\end{array}\right\} & \begin{array}{c}\Rightarrow\\ \textit{Rem. }15\\ \not\Leftarrow\\ \textit{Rem. }15\end{array} & (\bar{c},\bar{b})\in \operatorname{int}\left\{(c,b):\begin{array}{c}P(c,b)\text{ has a}\\ \text{strongly unique}\\ \text{minimizer}\end{array}\right\}\\
\textit{Thm. }10\Downarrow \quad \not\Uparrow\ \textit{Rem. }11 & & \not\Uparrow\ \textit{Rem. }15\\
\left\{\begin{array}{c}\mathcal{G}^*\text{is metrically regular}\\ \text{at } (\overline{x},(\bar{c},\bar{b}))\in gph\mathcal{G}^*\end{array}\right\} & \begin{array}{c}\Leftrightarrow\\ \textit{Lemma }5\end{array} & \left\{\begin{array}{c}\mathcal{F}^*\text{ is single-valued and Lipschitz}\\ \text{in a neighborhood of } (\overline{x},(\bar{c},\bar{b}))\\ \text{(strongly Lipschitz stable)}\end{array}\right\}\\
 & & \textit{obvious}\Downarrow \quad \not\Uparrow\ \textit{Rem. }17\\
 & & \left\{\begin{array}{c}\mathcal{F}^*\text{ is single-valued and continuous}\\ \text{in a neighborhood of } (\overline{x},(\bar{c},\bar{b}))\\ \text{(Kojima-type stability)}\end{array}\right\}
\end{array}
$$

When confined to the linear case, Theorem 16 establishes the equivalence among all of the conditions above.

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REFERENCES

[1] B. Bank, J. Guddat, D. Klatte, B. Kummer, and K. Tammer, *Nonlinear Parametric Optimization*, Birkhäuser Verlag, Basel, Boston, 1983.
[2] C. Berge, *Topological Spaces*, Macmillan, New York, 1963.
[3] B. Brosowski, *Parametric semi-infinite linear programming* I. *Continuity of the feasible set and of the optimal value*, Math. Programming Stud., 21 (1984), pp. 18–42.
[4] M. J. Cánovas, A. L. Dontchev, M. A. López, and J. Parra, *Metric regularity of semi-infinite constraint systems*, Math. Program. Ser. B, 104 (2005), pp. 329–346.
[5] M. J. Cánovas, M. A. López, J. Parra, and M. I. Todorov, *Stability and well-posedness in linear semi-infinite programming*, SIAM J. Optim., 10 (1999), pp. 82–98.
[6] A. L. Dontchev, A. S. Lewis, and R. T. Rockafellar, *The radius of metric regularity*, Trans. Amer. Math. Soc., 355 (2003), pp. 493–517.
[7] T. Fischer, *Contributions to semi-infinite linear optimization*, in Approximation and Optimization in Mathematical Physics, B. Brosowski and E. Martensen, eds., Peter Lang, Frankfurt-Am-Main, Germany, 1983, pp. 175–199.

[8] V. E. GAYÁ, M. A. LÓPEZ, AND V. N. VERA DE SERIO, *Stability in convex semi-infinite programming and rates of convergence of optimal solutions of discretized finite subproblems*, Optimization, 52 (2003), pp. 693–713.

[9] M. A. GOBERNA AND M. A. LÓPEZ, *Linear Semi-Infinite Optimization*, John Wiley & Sons, Chichester, UK, 1998.

[10] M. A. GOBERNA, M. A. LÓPEZ, AND M. I. TODOROV, *Stability theory for linear inequality systems*, SIAM J. Matrix Anal. Appl., 17 (1996), pp. 730–743.

[11] M. A. GOBERNA, M. A. LÓPEZ, AND M. I. TODOROV, *A generic result in linear semi-infinite optimization*, Appl. Math. Optim., 48 (2003), pp. 181–193.

[12] S. GOMEZ, A. LANCHO, AND M. TODOROV, *Stability in convex semi-infinite optimization*, C. R. Acad. Bulgare Sci., 55 (2002), pp. 23–26.

[13] S. HELBIG AND M. I. TODOROV, *Unicity results for general linear semi-infinite optimization problems using a new concept of active constraints*, Appl. Math. Optim., 38 (1998), pp. 21–43.

[14] R. HETTICH AND P. ZENCKE, *Numerische Methoden der Approximation und Semi-infinite Optimierung*, Teubner, Stuttgart, 1982.

[15] J.-B. HIRIART-URRUTY AND C. LEMARÉCHAL, *Convex Analysis and Minimization Algorithms* I, Springer-Verlag, Berlin, 1993.

[16] W. W. HOGAN, *Point-to-set maps in mathematical programming*, SIAM Rev., 15 (1973), pp. 591–603.

[17] D. KLATTE, *Stability of stationary solutions in semi-infinite optimization via the reduction approach*, in Advances in Optimization, Lecture Notes in Econom. and Math. Systems 382, W. Oettli and D. Pallaschke, eds., Springer-Verlag, Berlin, 1992, pp. 155–170.

[18] D. KLATTE AND B. KUMMER, *Stability properties of infima and optimal solutions of parametric optimization problems*, in Nondifferentiable Optimization: Motivations and Applications, V. F. Demyanov and D. Pallaschke, eds., Springer-Verlag, Berlin, 1985, pp. 215–229.

[19] D. KLATTE AND B. KUMMER, *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications*, Nonconvex Optim. Appl. 60, Kluwer Academic, Dordrecht, The Netherlands, 2002.

[20] D. KLATTE AND B. KUMMER, *Strong Lipschitz stability of stationary solutions for nonlinear programs and variational inequalities*, SIAM J. Optim., 16 (2005), pp. 96–119.

[21] M. KOJIMA, *Strongly stable stationary solutions in nonlinear programs*, in Analysis and Computation of Fixed Points, S. M. Robinson, ed., Academic Press, New York, 1980, pp. 93–138.

[22] G. NÜRNBERGER, *Unicity in semi-infinite optimization*, in Parametric Optimization and Approximation, B. Brosowski and F. Deutsch, eds., Birkhäuser, Basel, 1984, pp. 231–247.

[23] B. T. POLYAK, *Introduction to Optimization*, Optimization Software, New York, 1987.

[24] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[25] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[26] J.-J. RÜCKMANN, *On existence and uniqueness of stationary points in semi-infinite optimization*, Math. Programming, 86 (1999), pp. 387–415.